Multi-Sided Boxes

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1 Introduction

Multi-sided boxes are turned from cylindrical blanks using multi-axis turning. Figures 1 and 2 (see the last 3 pages for the figures of this document) illustrate two possible setups for 3-sided triangular boxes. In each of these figures the points numbered 1, 2, and 3 on the solid circle inside the blank are where the centers are placed for the three axes along which the outside of the triangular box will be formed. The shaded areas will be the portions of the blank removed in the turning process. The dashed circles in each of these figures give us the depth of the cuts we make in forming the outside of the box. Figure 3 illustrates a corresponding setup for a 4-sided square box and Figure 4 does likewise for a 6-sided hexagonal box. From now on the inside circle will be referred to as the **axes circle** and the dashed circle the **depth circle**.

One thing to notice in Figures 1 and 2 is that the larger the radius of the axes circle the straighter (or less curvature) of the sides of the box. This extends to boxes with more sides so that if axes circles of smaller radius had been used in Figures 3 or 4 the sides of the boxes would be rounder (or have more curvature). I prefer to have straighter sides so I want to have larger axes circles. But there is also a limit to their size. I use 1/2 inch diameter drive and live Sorby steb centers when turning my boxes so that if the radius of a depth circle in Figures 1-4 is less than 1/4 inch larger than the radius of the axes circle my steb center might not fully seat as I move from turning on one axis to another.

There is a formula for determining the radius r of the axes circle depending on the radius a of the cylindrical blank, the number of sides n of the box, the difference b of the radii of the depth circle and axes circle, and the angle $\theta = 360°/n$. (For illustration, see Figure 5 where the inner arc between the points D and E is on a side of the box, r is the distance between the points O and A, a is the distance between the points O and D, b is the distance between the points B and C, and θ is the angle formed by the points E, O, and D.) This formula is:

$$r = \frac{2a\cos(\theta/2) - 4b + \sqrt{4b^2 - 16ab\cos(\theta/2) + 4a^2\cos^2(\theta/2) + 12a^2}}{6}. \quad (1)$$

As mentioned in the previous paragraph, I use 1/2 inch steb centers so that $b = 1/4$ and for me this formula becomes

$$r = \frac{2a\cos(\theta/2) - 1 + \sqrt{1/4 - 4a\cos(\theta/2) + 4a^2\cos^2(\theta/2) + 12a^2}}{6}. \quad (2)$$

In the next section, Section 2, the formula in (2) will be used to obtain tables of values of r for values of a that I frequently use to turn triangular ($n = 3$) and hexagonal ($n = 6$) boxes. In Section 3 we will do likewise for square ($n = 4$) and octagonal ($n = 8$) boxes. Pentagonal ($n = 5$) boxes will be considered in Section 4. If you want to use, say, 5/8 inch centers you will have to do the calculations yourself using $b = 5/16$ in the formula in (1). If you are uncomfortable dealing with the trigonometric function cosine, don't completely despair—we'll see that it can be eliminated from the formulas for triangular, hexagonal, square, and octagonal boxes. However, it cannot be eliminated for pentagonal boxes.

The steps I use to turn a multi-sided box begin as follows:

Step 1. Start with a rectangular blank with square ends and end grain on the square ends. (I typically use blanks that are 3×3 to 4×4 inches on the square end and 4 to 5 inches long.) Turn the blank into a cylinder (with side grain parallel to lathe bed).

Step 2. Before removing the cylindrical blank from lathe, bring tool rest flush with blank and use it as a straight edge to draw a **reference line** along the side of the blank.

Step 3. Remove blank from lathe. On each end draw the **radial line** from the reference line to center of blank.

Step 4. On each end draw the axes circles and label point where radial line and axes circle intersect as position 1. Mark off equally spaced **axes points** from position 1 around the axes circle on each end numbering these points 2, 3, ... **clockwise on one end** and **counterclockwise on the other end**. Methods for marking off these axes points will be discussed in Sections 2-4. Then use an awl to make holes at the axes points.

After we discuss setups for triangular, hexagonal, square, octagonal, and pentagonal boxes in Sections 2-4, steps will be given for completing the turning of a box in Section 5. The derivation for the formula in (1) will be given in Section 6 for those interested in seeing how this is done. The methods used here are extendable to boxes with other numbers of sides such as 9-sided or nonagonal boxes, 10-sided or decagonal boxes, and so on. The more sides the more one is hitting air as the sides of the box are being turned resulting in rougher cuts as well as intersections of adjacent sides, which should be straight lines, being wavy. The last section, Section 7, is devoted to twisted multi-sided boxes.

2 Triangular and Hexagonal Boxes

For a triangular box $\theta = 360°/3 = 120°$ making $\theta/2 = 60°$. Here the cosine of $\theta/2$ is nice: $\cos 60° = 1/2$. Filling this into the formula in (2) and simplifying we end up with

$$r = \frac{a - 1 + \sqrt{1/4 - 2a + 13a^2}}{6}. \tag{3}$$

The following table gives some values of r for values of a that come up when I start with a blank 3 to 4 inches on the square end.

Triangular Box Table

a	$1\frac{3}{8}$	$1\frac{1}{2}$	$1\frac{5}{8}$	$1\frac{3}{4}$	$1\frac{7}{8}$
r	$\frac{13}{16}$	$\frac{15}{16}$	1	$1\frac{1}{8}$	$1\frac{3}{16}$

I round the values of a of my cylindrical blank down when using this table. For instance, if I start with a square blank with a 3×3 inch square end, the radius of the cylindrical blank I get is likely to be a bit under $1\frac{1}{2}$ inches so will use the value $a = 1\frac{3}{8}$ in the table. Rounding up could lead to the sides not meeting. I found the values of r in the table using a calculator (one with a square root key is required here) rounding down to the next sixteenth of an inch. For instance, when $a = 1\frac{3}{8}$ my calculator gives me 0.85622755 as the approximate value of r which I rounded to $\frac{13}{16} = 0.8125$ (and did not use $\frac{7}{8} = 0.875$ to insure the sides meeting). The same rounding down procedure will be followed for boxes with more than three sides.

Marking off the axes points on the axes circle in Step 4 is easy for triangular boxes: Leaving your compass in the same setting you used when drawing the axes circle, place the point of your compass at position 1 and make a first mark with your compass in the clockwise direction on the axes circle, then place the point of your compass at this first mark, make a second mark in the clockwise direction and so on. As is illustrated in Figure 6, after six such steps you come back to position 1. Using every other mark gives us the desired three axes points on one end of the cylindrical blank. Do likewise on the other end remembering to number counterclockwise.

Once the radius of the axes circle is determined for hexagonal boxes, the same process gives us the six required axes points around the axes circle—simply use every mark as illustrated in Figure 7. For a hexagonal box $\theta = 360°/6 = 60°$ making $\theta/2 = 30°$. As $\cos 30° = \sqrt{3}/2$ substituting this value into the formula in (2) and simplifying yields the formula

$$r = \frac{\sqrt{3}a - 1 + \sqrt{1/4 - 2\sqrt{3}a + 15a^2}}{6}. \tag{4}$$

Using this formula we get the following table for hexagonal boxes:

Hexagonal Box Table

a	$1\frac{3}{8}$	$1\frac{1}{2}$	$1\frac{5}{8}$	$1\frac{3}{4}$	$1\frac{7}{8}$
r	1	$1\frac{1}{8}$	$1\frac{1}{4}$	$1\frac{3}{8}$	$1\frac{1}{2}$

3 Square and Octagonal Boxes

For a square box $\theta = 360°/4 = 90°$ making $\theta/2 = 45°$. As $\cos 45° = \sqrt{2}/2$ substituting this value into the formula in (2) and simplifying yields the formula

$$r = \frac{\sqrt{2}a - 1 + \sqrt{1/4 - 2\sqrt{2}a + 14a^2}}{6} \tag{5}$$

from which we obtain the table:

Square Box Table

a	$1\frac{3}{8}$	$1\frac{1}{2}$	$1\frac{5}{8}$	$1\frac{3}{4}$	$1\frac{7}{8}$
r	$\frac{15}{16}$	1	$1\frac{3}{16}$	$1\frac{1}{4}$	$1\frac{3}{8}$

To determine the four axes points around the axes circle in Figure 8, I extended the radial line through the circular end of the cylindrical blank and then used my compass and a straightedge to construct the line perpendicular to the radial line at the center of the blank. Of course, you might rather construct this perpendicular line using a square.

Once the radius of the axes circle is determined for octagonal boxes, the eight required axes points around the axes circle are obtained in Figure 9 by repeating the work of Figure 8 and then bisecting the right angles between points 1 and 2 and between points 2 and 3 in Figure 8 with a straightedge and compass. Other ways would be to use a protractor or something with a 45° angle such as a piece of wood cut at 45° to obtain the additional four axes points needed on the axes circle to get Figure 9 from Figure 8. In an octagonal box $\theta = 360°/8 = 45°$ making $\theta/2 = 22.5°$ so that the formula in (2) is

$$r = \frac{2a\cos 22.5° - 1 + \sqrt{1/4 - 4a\cos 22.5° + 4a^2\cos^2 22.5° + 12a^2}}{6} \qquad (6)$$

for an octagonal box. The value of $\cos 22.5°$ is $\sqrt{2+\sqrt{2}}/2$. Substituting this value into the formula in (6) and simplifying yields the formula

$$r = \frac{\sqrt{2+\sqrt{2}}a - 1 + \sqrt{1/4 - 2\sqrt{2+\sqrt{2}}a + 14a^2 + \sqrt{2}a^2}}{6}. \qquad (7)$$

Of theses two formulas I find the formula in (6) more convenient to use. Of course, you may use the formula in (7) if you want to avoid using the cosine function. Using either formula we get the following table:

Octagonal Box Table

a	$1\frac{3}{8}$	$1\frac{1}{2}$	$1\frac{5}{8}$	$1\frac{3}{4}$	$1\frac{7}{8}$
r	$1\frac{1}{16}$	$1\frac{3}{16}$	$1\frac{1}{4}$	$1\frac{7}{16}$	$1\frac{1}{2}$

4 Pentagonal Boxes

In a pentagonal box $\theta = 360°/5 = 72°$ making $\theta/2 = 36°$ so that the formula in (2) is

$$r = \frac{2a\cos 36° - 1 + \sqrt{1/4 - 4a\cos 36° + 4a^2\cos^2 36° + 12a^2}}{6} \qquad (8)$$

for these boxes. Here $\cos 36°$ does not have an easily expressible value as with our previous angles so that using the formula in (8) is our only option. Using it we get the following table:

Pentagonal Box Table

a	$1\frac{3}{8}$	$1\frac{1}{2}$	$1\frac{5}{8}$	$1\frac{3}{4}$	$1\frac{7}{8}$
r	1	$1\frac{1}{16}$	$1\frac{3}{16}$	$1\frac{5}{16}$	$1\frac{7}{16}$

The five required axes points around the axes circle can be obtained in Figure 10 by using protractor to lay out 5 consecutive 72° angles going around the circle. An alternative would be to use a piece of wood cut with a 72° angle to lay out these angles.

5 Remaining Steps

The remaining steps I use to turn a multi-sided box are as follows:

Step 5. Draw depth circles with radius 1/4 inch larger than the radius of axes circle on each end.

Step 6. Mount cylindrical blank on center on lathe and turn tenons on each end to depth circles.

Step 7. Remount blank on lathe at position 1 on axes circles at each end. Turn down to tenons. Repeat for remaining positions on axes circles.

Step 8. Remount piece on center and sand sides. Turn tenons on ends deeper later for chucking.

Step 9. Chuck piece at what will be base of box. Begin process of parting off the lid of the box (I leave about $1\frac{1}{4}$ inches for the lid) by making opposing pommel cuts about 7/16 inches apart. Complete parting off lid between pommel cuts at top end of base of box.

Step 10. Drill out inside of base with forstner bit. If desired, turn out divot from drill bit on base of box. Sand inside and top of base.

Step 11. Replace base by lid in chuck. Turn section below pommel cut to fit base. Complete turning bottom of lid and sand.

Step 12. Replace lid by base in chuck. Use base as jam chuck to complete top of lid. Bring up tailstock, make pommel cut on top of lid, round lid as far as possible, and sand. Tape lid to base, remove tailstock, and complete rounding and sanding lid.

Step 13. Remove lid from base and base from chuck. Chuck in a jam chuck to complete bottom of base. Turn jam chuck to fit inside of base, put base on jam chuck, and bring up tailstock. Make pommel cut at bottom of base and remove as much of tenon on base as possible. Tape base to jam chuck. Then complete and sand bottom of base.

You've now completed turning your box. Remove base from jam chuck and finish box as desired.

6 Derivation of the Formula

To derive the formula in (1) consider ΔAOD in Figure 5. $\angle COD$ is $\theta/2$ so that $\angle AOD$ is $180° - \theta/2$. Applying the law of cosines at $\angle AOD$ to ΔAOD we get

$$(b+2r)^2 = r^2 + a^2 - 2ar\cos(180° - \theta/2).$$

Since $\cos(180° - \theta/2) = -\cos(\theta/2)$ this becomes

$$(b+2r)^2 = r^2 + a^2 + 2ar\cos(\theta/2)$$

or

$$b^2 + 4br + 4r^2 = r^2 + a^2 + 2ar\cos(\theta/2).$$

Rearranging the previous equation in the form

$$3r^2 + (4b - 2a\cos(\theta/2))r + b^2 - a^2 = 0$$

and applying the quadratic formula to solve for r we get

$$r = \frac{-(4b - 2a\cos(\theta/2)) \pm \sqrt{(4b - 2a\cos(\theta/2))^2 - 12(b^2 - a^2)}}{6}$$

which becomes

$$r = \frac{2a\cos(\theta/2) - 4b \pm \sqrt{4b^2 - 16ab\cos(\theta/2) + 4a^2\cos^2(\theta/2) + 12a^2}}{6}.$$

Because $b < a$ the expression under the square root is greater than $16b^2 - 16ab\cos(\theta/2) + 4a^2\cos(\theta/2)$ which is the square of $2a\cos(\theta/2) - 4b$. This means we will get a negative value for r if we use the negative square root. Consequently we must use the positive square root which is the formula in (1).

7 Twisted Multi-Sided Boxes

Twisted multi-sided boxes are obtained by rotating the axes points on one end of the cylindrical blank. For example, in a triangular box one might move position 1 to position 2, position 2 to position 3, and position 3 to position 1 on one end which amounts to a 120° rotation. As another example, in a square box one might move position 1 to position 2, position 2 to position 3, position 3 to position 4, and position 4 to position 1 on one end which amounts to a 90° rotation. These a just a couple of examples and the rotations do not necessarily require one position be moved to the next. For instance, in a triangular box one could rotate each position over 50°. Also, whether the rotation is in the counterclockwise or clockwise direction is immaterial in the sense that a counterclockwise rotation on one end is the same as a clockwise rotation on the other end and vice-versa. The formula in (2) does not work for twisted boxes in that the sides of the boxes will not meet in the center. This is caused by the turning axes being closer to the center of the cylindrical blank in the middle of the blank. Just how much closer will be described later. Of course, one can simply make the box with the sides not meeting in the center leaving part of the original cylinder of the blank in the sides of the box which is itself an interesting feature. But what do we do if we do want the sides to meet? Using ρ for the angle of the rotation (with a and $\theta = 360°/n$ as before and with 1/2 inch drive and live centers), I use the following formula to calculate the radius of the axes circle:

$$r = \frac{p}{q} \tag{9}$$

where

$$p = 2a\cos(\rho/2)\cos(\theta/2) - 1 +$$
$$\sqrt{\frac{1}{4}\cos^2(\rho/2) + 16a^2 - 4a^2\cos^2(\rho/2)\sin^2(\theta/2) - 4a\cos(\rho/2)\cos(\theta/2)}$$

and

$$q = 8 - 2\cos^2(\rho/2).$$

This is not the best possible value of r in the sense (as will be discussed later) there is a larger value of r that can be used, but for me the formula given in (9) is more convenient to use. Once the value of r is determined, the steps for turning the box are the same as the earlier ones.

We next consider twists of moving one position to the next on one end for the boxes we considered earlier. In the case of a triangular box with a 120° twist or rotation $\cos(\rho/2) = \cos(\theta/2) = \cos 60^\circ = 1/2$, $\sin(\theta/2) = \sin 60^\circ = \sqrt{3}/2$, and the formula in (9) becomes

$$r = \frac{a - 2 + 2\sqrt{1/16 + 61a^2/4 - a}}{15}.$$

Using it we get a table corresponding to the earlier triangular box table:

120° Twist Triangular Box Table

a	$1\frac{3}{8}$	$1\frac{1}{2}$	$1\frac{5}{8}$	$1\frac{3}{4}$	$1\frac{7}{8}$
r	$\frac{5}{8}$	$\frac{11}{16}$ or $\frac{3}{4}$	$\frac{3}{4}$ or $\frac{13}{16}$	$\frac{7}{8}$	$\frac{15}{16}$

In constructing this table I sometimes made exceptions to the convention of rounding down the values of r to the next sixteenth of an inch. Because the formula does not give the largest possible r, it is probably safe to instead round to the nearest sixteenth of an inch. For instance, when $a = 1\frac{1}{2}$ my calculator gives me 0.731156329 for r and I previously would have used $r = \frac{11}{16} = 0.6875$. However, rounding r up to $\frac{3}{4} = 0.75$ probably will still yield the sides fully meeting; if not, further turning of the sides with slightly deeper cuts hopefully will get them to meet while still allowing for sufficient contact of the centers. To reflect this option I have put both the values $\frac{11}{16}$ and $\frac{3}{4}$ in the table for r when $a = 1\frac{1}{2}$. Likewise, for $a = 1\frac{5}{8}$ where my calculator gives me 0.804533001 for r rounding up to $\frac{13}{16} = 0.8125$ will probably work okay so I put $\frac{3}{4}$ or $\frac{13}{16}$ in the table. This new convention will be followed in the tables for the remaining twisted boxes given here.

For a hexagonal box with a 60° twist or rotation $\cos(\rho/2) = \cos(\theta/2) = \cos 30^\circ = \sqrt{3}/2$, $\sin(\theta/2) = \sin 30^\circ = 1/2$, and the formula in (9) becomes

$$r = \frac{3a - 2 + 2\sqrt{3/16 + 61a^2/4 - 3a}}{13}$$

from which we get the table:

60° Twist Hexagonal Box Table

a	$1\frac{3}{8}$	$1\frac{1}{2}$	$1\frac{5}{8}$	$1\frac{3}{4}$	$1\frac{7}{8}$
r	$\frac{7}{8}$ or $\frac{15}{16}$	1 or $1\frac{1}{16}$	$1\frac{1}{8}$	$1\frac{3}{16}$ or $1\frac{1}{4}$	$1\frac{5}{16}$ or $1\frac{3}{8}$

For a square box with a 90° twist or rotation $\cos(\rho/2) = \cos(\theta/2) = \cos 45^\circ = \sqrt{2}/2$, $\sin(\theta/2) = \sin 45^\circ = \sqrt{2}/2$, and the formula in (9) becomes

$$r = \frac{a - 1 + 2\sqrt{1/8 + 15a^2 - 2a}}{7}$$

from which we get the table:

90° Twist Square Box Table

a	$1\frac{3}{8}$	$1\frac{1}{2}$	$1\frac{5}{8}$	$1\frac{3}{4}$	$1\frac{7}{8}$
r	$\frac{3}{4}$	$\frac{13}{16}$ or $\frac{7}{8}$	$\frac{15}{16}$	1 or $1\frac{1}{16}$	$1\frac{1}{8}$

For an octagonal box with a 45° twist or rotation $\cos(\rho/2) = \cos(\theta/2) = \cos 22.5°$, $\sin(\theta/2) = \sin 22.5°$, and (not filling in the values of the trig functions) the formula in (9) becomes

$$r = \frac{2a\cos^2 22.5° - 1 + \sqrt{\frac{\cos^2 22.5°}{4} + 16a^2 - 4a^2\cos^2 22.5°\sin^2 22.5° - 4a\cos^2 22.5°}}{8 - 2\cos^2 22.5°}$$

from which we get the table:

45° Twist Octagonal Box Table

a	$1\frac{3}{8}$	$1\frac{1}{2}$	$1\frac{5}{8}$	$1\frac{3}{4}$	$1\frac{7}{8}$
r	1	$1\frac{1}{16}$ or$1\frac{1}{8}$	$1\frac{3}{16}$ or $1\frac{1}{4}$	$1\frac{5}{16}$	$1\frac{7}{16}$

For an pentagonal box with a 72° twist or rotation $\cos(\rho/2) = \cos(\theta/2) = \cos 36°$, $\sin(\theta/2) = \sin 36°$, and the formula in (9) becomes

$$r = \frac{2a\cos^2 36° - 1 + \sqrt{\frac{\cos^2 36°}{4} + 16a^2 - 4a^2\cos^2 36°\sin^2 36° - 4a\cos^2 36°}}{8 - 2\cos^2 36°}$$

from which we get the table:

72° Twist Pentagonal Box Table

a	$1\frac{3}{8}$	$1\frac{1}{2}$	$1\frac{5}{8}$	$1\frac{3}{4}$	$1\frac{7}{8}$
r	$\frac{13}{16}$ or $\frac{7}{8}$	$\frac{15}{16}$	1 or $1\frac{1}{16}$	$1\frac{1}{8}$ or$1\frac{3}{16}$	$1\frac{1}{4}$

Of course, many more such tables for twisted multi-sided boxes can be calculated that will be left to those desiring them.

We conclude with a discussion sketching out the main details of how r is determined from a for twisted multi-sided boxes for those interested in getting a sense of this. Knowledge about the Cartesian coordinate system and vectors in 3-dimensions will be required. In Figure 11 the larger cylinder is the cylindrical blank of radius a and h is its height. The inner cylinder is the one of radius r formed from the axes circle, the point P is one of the axis points (placed for convenience on the x-axis), Q is the point on the opposite end of the blank rotated through the angle ρ, R is the midpoint of the center axis of the cylinders, M is the midpoint between P and Q, and s is the distance between R and M. Using the distance formula for points in 3-space, this distance is

$$s = \sqrt{(\frac{1}{2}r\cos\rho + \frac{1}{2}r - 0)^2 + (\frac{1}{2}r\sin\rho - 0)^2 + (\frac{h}{2} - \frac{h}{2})^2}$$

which simplifies to

$$s = r\sqrt{\frac{1+\cos\rho}{2}}.$$

Because all possible rotations can be obtained with $-180° \leq \rho \leq 180°$ where a negative value of ρ is a clockwise rotation and a positive value of ρ is a counterclockwise rotation (in fact, we can actually restrict to $0 \leq \rho \leq 180°$ by the comment about only needing counterclockwise rotations in the first paragraph of this section) we can use the positive square root in the half angle formula $\cos(\rho/2) = \pm\sqrt{(1+\cos\rho)/2}$ so that the preceding equation becomes

$$s = r\cos(\rho/2). \tag{10}$$

Equation (10) then tells how closer the turning axis is to the middle of the blank. For instance, when $\rho = 120°$ we have $s = r/2$ so it has moved halfway in; when $\rho = 90°$ we have $s = \sqrt{2}r/2$ which is about $0.7r$ so it has moved about three-tenths of the way in.

In Figure 12 the points V and W are on the side of the cylindrical blank and the arc between them is the portion of the circle forming the side of the box on the plane perpendicular to the line through the points P and Q at the point M in Figure 11. Also in Figure 12, d is the radius of the circular arc, U is the point that is directly under V on both the side of the cylindrical blank and the horizontal plane through the point M. The points M, U, and V then form a right triangle with right angle at U where the angle at M is labeled α and f is the distance between M and U.

Figure 13 illustrates the triangle formed by the points M, U, and R (from Figures 11 and 12) in the horizontal plane through M where the indicated exterior angle at R is necessarily $\theta/2$. Using the law of cosines we get

$$f^2 = a^2 + s^s - 2as\cos(180° - \theta/2) = a^2 + s^2 + 2as\cos(\theta/2)$$

from which equation (10) gives us

$$f = \sqrt{a^2 + r^2\cos^2(\rho/2) + 2ar\cos(\rho/2)\cos(\theta/2)}. \tag{11}$$

We have that

$$d = f\sec\alpha.$$

Also, the angle α is the same as the angle between the vector $\mathbf{v}$ from the point P to the point Q in Figure 11 which is $\mathbf{v} = (r\cos\rho - r)\mathbf{i} + r\sin\rho\mathbf{j} + h\mathbf{k}$ and the vector $\mathbf{k}$. Since $\mathbf{v}\cdot\mathbf{k} = |\mathbf{v}||\mathbf{k}|\cos\alpha$ where $\mathbf{v}\cdot\mathbf{k}$ is the dot product of $\mathbf{v}$ and $\mathbf{k}$ we have

$$h = \sqrt{(r\cos\rho - r)^2 + r^2\sin^2\rho + h^2}\cos\alpha$$

which yields

$$\cos\alpha = \frac{h}{\sqrt{2r^2 - 2r^2\cos\rho + h^2}}.$$

Since $\sec\alpha = 1/\cos\alpha$ we then have

$$d = \sqrt{a^2 + r^2\cos^2(\rho/2) + 2ar\cos(\rho/2)\cos(\theta/2)} \cdot \frac{\sqrt{2r^2 - 2r^2\cos\rho + h^2}}{h}.$$

Because d is the depth to which we must cut from the axis through the points P and Q and, at the ends of the blank, this distance is $2r + 1/4$ we have the equation

$$2r + \frac{1}{4} = \frac{\sqrt{a^2 + r^2\cos^2(\rho/2) + 2ar\cos(\rho/2)\cos(\theta/2)}\sqrt{2r^2 - 2r^2\cos\rho + h^2}}{h}$$

or

$$2hr + \frac{1}{4}h =$$
$$\sqrt{a^2 + r^2\cos^2(\rho/2) + 2ar\cos(\rho/2)\cos(\theta/2)}\sqrt{2r^2 - 2r^2\cos\rho + h^2}. \tag{12}$$

Squaring each side of this equation yields a 4^{th} degree polynomial or quartic equation in r. While there are methods for solving such equations (Wikipedia has an article on quartic equations), they are quite involved and I am going to avoid this. Instead of equating $2r + 1/4$ and d, I am going to equate $2r + 1/4$ and f from (11):

$$2r + \frac{1}{4} = \sqrt{a^2 + r^2\cos^2(\rho/2) + 2ar\cos(\rho/2)\cos(\theta/2)}.$$

While this yields a smaller value of r than necessary, the resulting equation is much easier to solve. Squaring each side we obtain

$$4r^2 + r + \frac{1}{16} = a^2 + r^2 \cos^2(\rho/2) + 2ar\cos(\rho/2)\cos(\theta/2)$$

or

$$(4 - \cos^2(\rho/2))r^2 + (1 - 2a\cos(\rho/2)\cos(\theta/2))r + \frac{1}{16} - a^2 = 0.$$

The formula in (9) is now obtained by applying the quadratic formula to the preceding equation. Finally, as h increases the angle α gets smaller so that $\sec\alpha$ gets closer to 1 which means that the formula in (9) yields values closer to the exact solution we would get from equation (12) the larger the value of h.

FIGURE 1

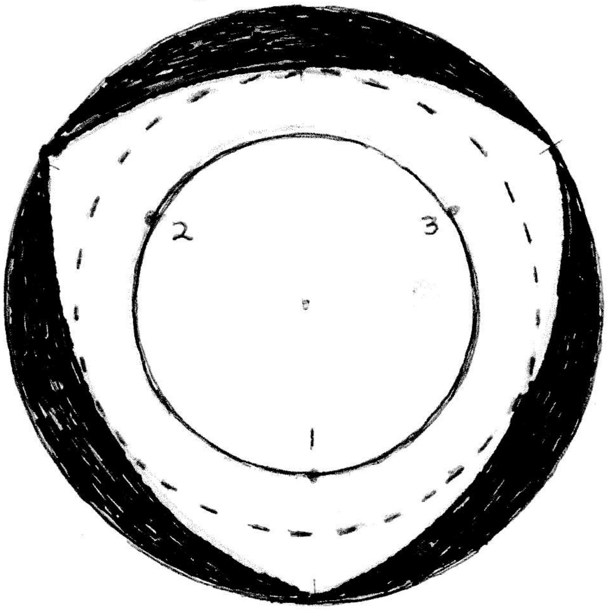

FIGURE 2

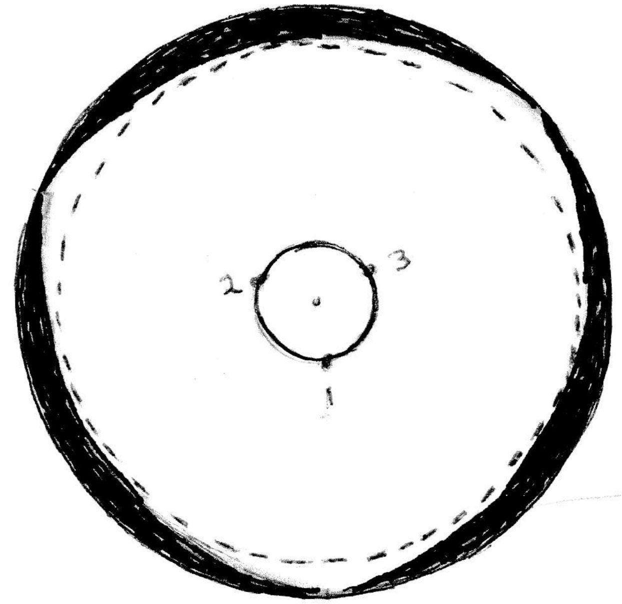

FIGURE 3

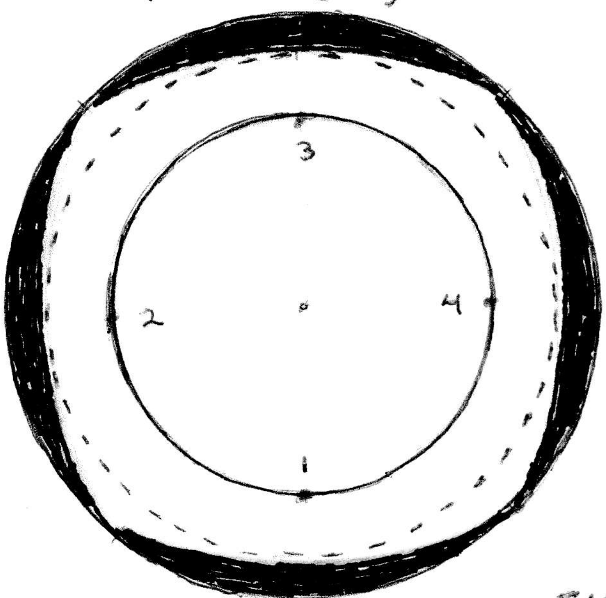

FIGURE 4

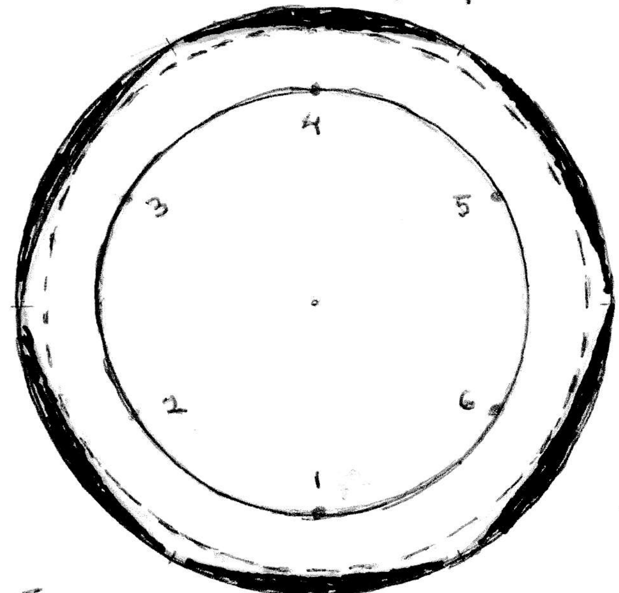

FIGURE 5

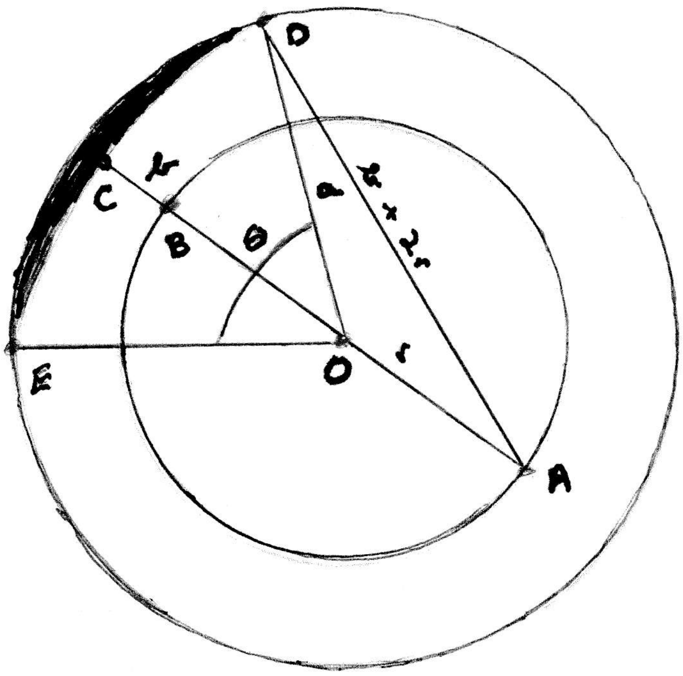

FIGURE 6

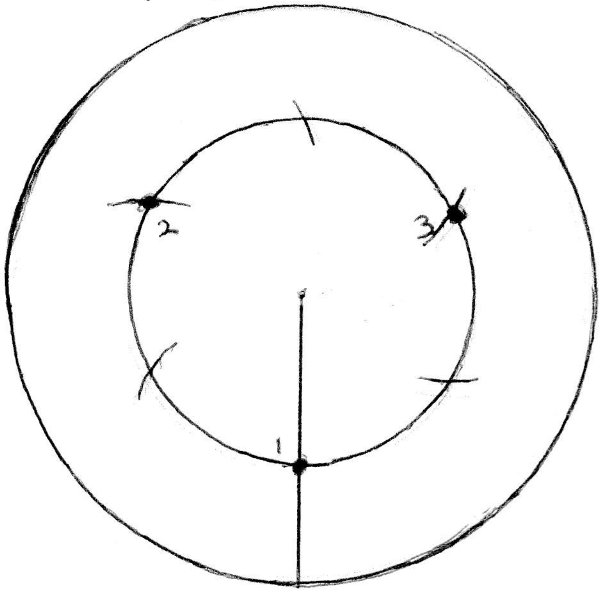

FIGURE 7

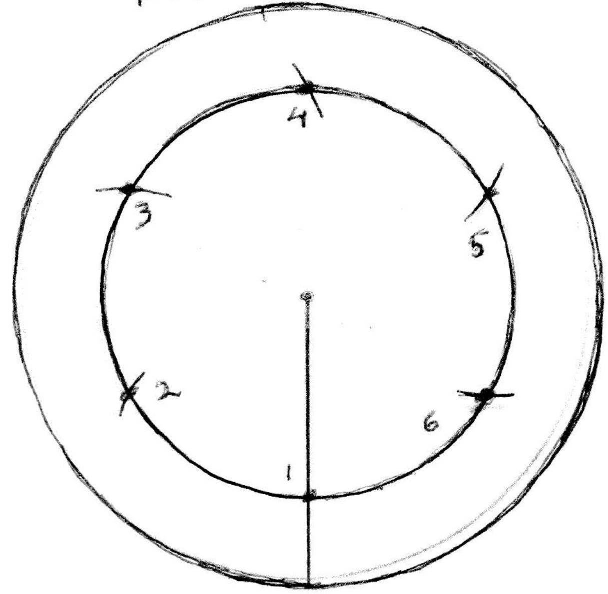

FIGURE 8

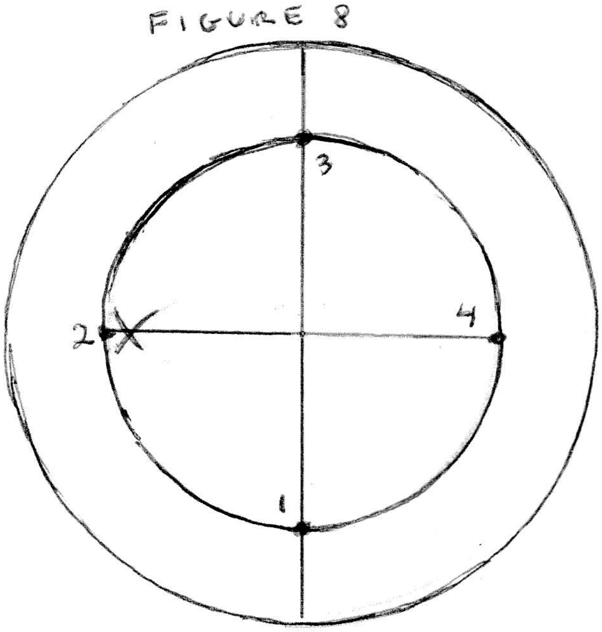

FIGURE 9

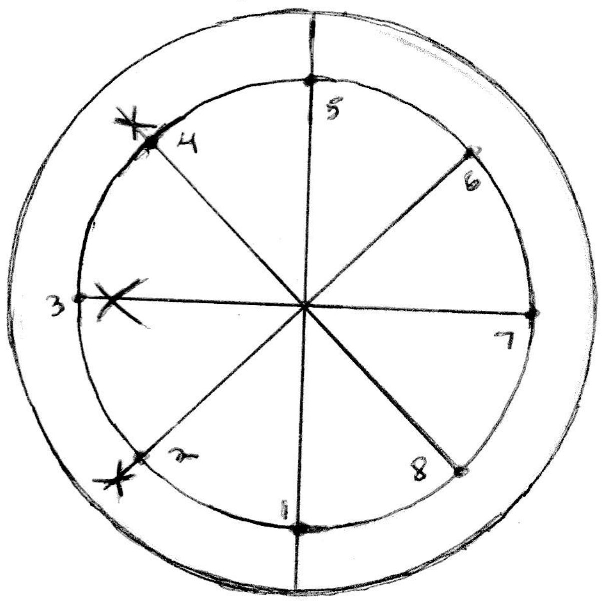

FIGURE 10

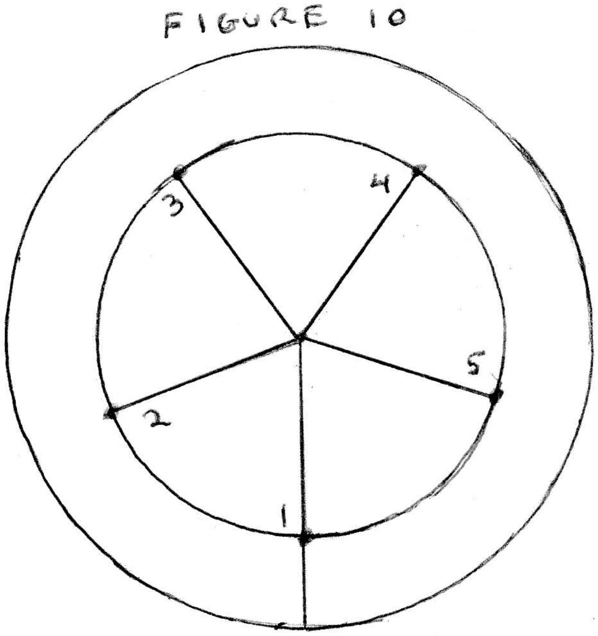

FIGURE 11

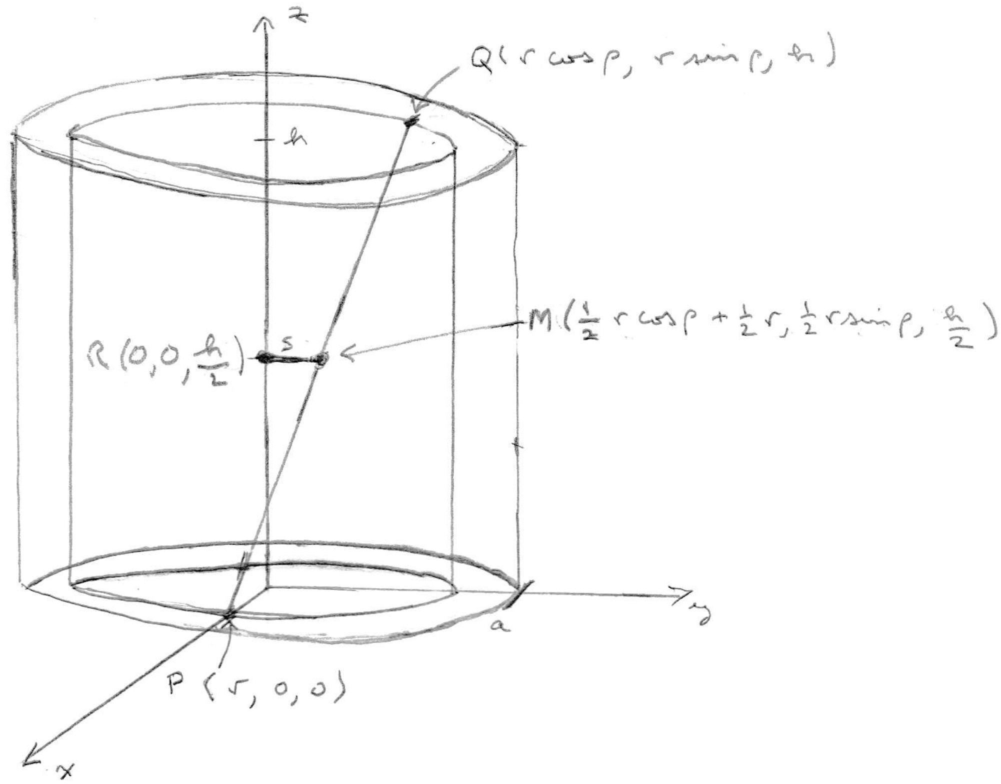

FIGURE 12

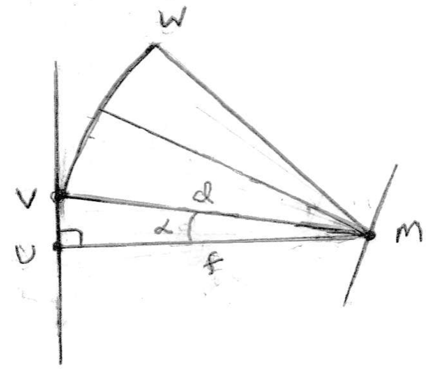

FIGURE 13